Math 431 - Real Analysis I
Solutions to Homework due October 22

Question 1. Sequences are frequently given *recursively*, where a beginning term x_1 is specified and subsequent terms can be found using a recursive relation. One such example is the sequence defined by $x_1 = 1$ and

$$x_{n+1} = \sqrt{2 + x_n}.$$

(a) For $n = 1, 2, \ldots, 10$, compute x_n. A calculator may be helpful.

(b) Show that x_n is a monotone increasing sequence. A proof by induction might be easiest.

(c) Show that the sequence x_n is bounded below by 1 and above by 2.

(d) Use (b) and (c) to conclude that x_n converges.

Solution 1.

(a)

n	x_n
1	1
2	1.41421
3	1.84776
4	1.96157
5	1.99036
6	1.99759
7	1.99939
8	1.99985
9	1.99996
10	1.99999

(b) Let $A(n)$ be the statement that $x_{n+1} \geq x_n$. We will show that $A(n)$ is true for all $n \geq 1$. First, note that $x_1 = 1 < \sqrt{3} = x_2$. Thus, $A(1)$ holds true. Now, assume that $A(k)$ is true. We will show that $A(k+1)$ is true. Thus, we will show that $x_{k+2} \geq x_{k+1}$. Notice that since $A(k)$ is true, then $x_{k+1} \geq x_k$. Thus, $2 + x_{k+1} \geq 2 + x_{k+2}$. Thus,

$$x_{k+2} = \sqrt{2 + x_{k+1}} \geq \sqrt{2 + x_k} = x_{k+1}.$$

Thus, $A(k+1)$ is true. So, by induction, $A(n)$ is true for all n.

(c) First, note that since $x_1 = 1$ and that x_n is monotone increasing, then $x_n \geq 1$ for all n. For the other bound, we will use induction on the statement $A(n)$ given by $x_n \leq 2$ for $n \geq 1$. For the base case, notice that $x_1 = 1 < 2$. Thus, $A(1)$ holds. Now, assume that $A(k)$ holds. WE will show that $A(k+1)$ is true. Since $A(k)$ is true, then $x_k \leq 2$. Thus,

$$x_{k+1} = \sqrt{2 + x_k} \leq \sqrt{2 + 2} = 2.$$

Thus, $A(k+1)$ is true. So, by induction, x_k is bounded above by 2.

(d) By (b) and (c), x_n is a bounded, monotone sequence; thus, by a theorem in class, it converges.

Question 2. One very important class of sequences are *series*, in which we add up the terms of a given sequence. One such example is the following sequence:

$$S_n = \sum_{k=0}^{n} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}.$$

(a) For $n = 0, 1, 2, \ldots, 6$, compute S_n. Again, a calculator may be helpful; be sure to use several digits.

(b) Show that S_n is monotone increasing.

(c) Use induction to show that for all $n \geq 1$, $n! \geq 2^{n-1}$.

(d) Use (c) to show that

$$S_n \leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}}.$$

(e) Use well-known facts from Calculus II and the geometric series to show that

$$1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} < 3.$$

(f) Use (b), (d), and (e) to conclude that S_n converges.

Solution 2.

(a)

n	S_n
0	1
1	2
2	2.5
3	2.66666
4	2.708333
5	2.7166666
6	2.7180555

(b) Since $1/n! > 0$, then

$$S_{n+1} = S_n + \frac{1}{(n+1)!} \geq S_n.$$

Thus, S_n is monotone increasing.

(c) Let $A(n)$ be the statement that $n! > 2^{n-1}$. We will show that $A(n)$ is true for all $n \geq 1$. For the base case, notice that $1! = 1 \geq 1 = 2^{1-1}$. Thus, $A(1)$ is true. Assume that $A(k)$ holds. Thus, $k! > 2^{k-1}$. We will show that $A(k+1)$ is true by showing that $(k+1)! < 2^k$. Notice that

$$(k+1)! = (k+1) \cdot k! \geq (k+1) \cdot 2^{k-1}.$$

Since $k \geq 1$, then $k+1 \geq 2$. So, $(k+1)2^{k-1} \geq 2 \cdot 2^{k-1} = 2^k$. Thus, $A(k+1)$ is true. So, $n! \geq 2^{n-1}$ for all $n \geq 1$.

(d) Since $n! \geq 2^{n-1}$, then we have that $\dfrac{1}{2^{n-1}} \geq \dfrac{1}{n!}$ for all $n \geq 1$. Comparing term-by-term, we have that

$$\sum_{k=1}^{n} \frac{1}{k!} \leq \sum_{k=1}^{n} \frac{1}{2^{k-1}}.$$

Since $\frac{1}{0!} = 1$, we have that

$$S_n = \sum_{k=0}^{n} \frac{1}{k!} \leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}}.$$

(e) From Calculus II, we recognize

$$\sum_{k=1}^{n} \frac{1}{2^{k-1}} = \sum_{j=0}^{n-1} \frac{1}{2^j} = \sum_{j=0}^{n-1} \left(\frac{1}{2}\right)^j$$

as a geometric series. Thus,

$$\sum_{j=0}^{n-1} \left(\frac{1}{2}\right)^j = \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} < \frac{1}{1-\frac{1}{2}} = 2.$$

Thus,

$$1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} < 1 + 2 = 3.$$

(f) By (b), S_n is a monotone sequence. By (d) and (e), we have that $S_n < 3$ and is thus bounded. Thus, since every bounded monotone sequence converges, S_n converges.

(g)

Question 3. In class, we learned that a sequence in $\mathbb{R}^k$ is convergent if and only if it is Cauchy. We have previously proven using the definition of convergence that the sequence

$$x_n = \frac{1}{n}$$

converges (to 0). Thus, it should also be Cauchy. In this problem, we will prove directly that it is Cauchy.

(a) Let $n, m \in \mathbb{Z}_+$. Show that

$$\left|\frac{1}{n} - \frac{1}{m}\right| < \frac{1}{n} + \frac{1}{m}.$$

(b) Use (a) to show that $x_n = \frac{1}{n}$ is a Cauchy sequence. To do so, given an $\varepsilon > 0$, find an N such that for all $n, m > N$,

$$\left|\frac{1}{n} - \frac{1}{m}\right| < \frac{1}{n} + \frac{1}{m} < \varepsilon.$$

Solution 3.

(a) We will show this inequality by showing that

$$-\frac{1}{n} - \frac{1}{m} < \frac{1}{n} - \frac{1}{m} < \frac{1}{n} + \frac{1}{m}.$$

For the second inequality, notice that since $\frac{-1}{m} < \frac{1}{m}$, then

$$\frac{1}{n} - \frac{1}{m} < \frac{1}{n} + \frac{1}{m}.$$

Similarly, it's clear that $\frac{-1}{n} < \frac{1}{n}$,, so we get that

$$-\frac{1}{n} - \frac{1}{m} < \frac{1}{n} - \frac{1}{m}.$$

Combining this gives the two inequalities, which is equivalent to

$$\left|\frac{1}{n} - \frac{1}{m}\right| < \frac{1}{n} + \frac{1}{m}.$$

(b) Let $\varepsilon > 0$. By the Archimedean principle, there exists an N such that $N\frac{\varepsilon}{2} > 1$. Thus, $\frac{1}{N} < \frac{\varepsilon}{2}$. Thus, for all $n, m > N$, we have that

$$\frac{1}{n}, \frac{1}{m} < \frac{1}{N} < \frac{\varepsilon}{2}.$$

Thus, by (a),

$$|x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| < \frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $x_n = \frac{1}{n}$ is a Cauchy sequence.

Question 4. Consider the sequence of partial sums given by

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2}.$$

We will show that S_n converges by showing it is Cauchy.

(a) If $n, m \in \mathbb{Z}_+$ with $m > n$, show that

$$|S_m - S_n| = \sum_{k=n+1}^{m} \frac{1}{k^2}.$$

(b) Show that $\frac{1}{k^2} < \frac{1}{k(k-1)}$ for $k \geq 2$.

(c) Show that

$$\sum_{k=n+1}^{m} \frac{1}{k(k-1)} = \frac{1}{n} - \frac{1}{m}.$$

As a hint, think about *telescoping series* from Calculus II.

(d) Use the above to show that

$$|S_m - S_n| < \frac{1}{m} + \frac{1}{n}.$$

(e) Use (d) in a proof to show that S_n is Cauchy and thus converges.

Solution 4.

(a) Since all the terms in the sum are positive and $m > n$, then $|S_m - S_n| = S_m - S_n$. The terms in $S_m - S_n$ are those terms up to m excluding the first n. Thus, we have that

$$|S_m - S_n| = \sum_{k=n+1}^{m} \frac{1}{k^2}.$$

(b) Notice that $k(k-1) = k^2 - k < k^2$. Since all terms are positive, we can cross-divide to get

$$\frac{1}{k^2} < \frac{1}{k(k-1)}.$$

(c) Notice that

$$\frac{1}{k(k-1)} = \frac{1}{k} - \frac{1}{k-1}.$$

Thus, when adding up the terms in

$$\sum_{k=n+1}^{m} \frac{1}{k(k-1)} = \sum_{k=n+1}^{m} \frac{1}{k} - \frac{1}{k-1},$$

all cancel except for the term $\frac{1}{n}$ and $-\frac{1}{m}$. Thus,

$$\sum_{k=n+1}^{m} \frac{1}{k(k-1)} = \frac{1}{n} - \frac{1}{m}.$$

(d) Putting the above together, we have that

$$|S_m - S_n| = \sum_{k=n+1}^{m} \frac{1}{k^2} < \sum_{k=n+1}^{m} \frac{1}{k(k-1)} = \frac{1}{m} - \frac{1}{n} < \frac{1}{m} + \frac{1}{n}.$$

(e) Let $\varepsilon > 0$. By the Archimedean property, there exists an $N \in \mathbb{Z}_+$ such that $N\frac{\varepsilon}{2} > 1$. Thus,

$$\frac{1}{N} < \frac{\varepsilon}{2}.$$

Question 5. Let $f, g : \mathbb{R} \to \mathbb{R}$ with $a, L, M, k \in \mathbb{R}$. Furthermore, assume that

$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = M.$$

Give an ε−δ proof to show the following:

(a) $\lim\limits_{x\to a} k \cdot f(x) = k \cdot L$

(b) $\lim\limits_{x\to a} f(x) + g(x) = L + M$

Solution 5.

(a) We will prove this in cases: $k = 0$ or $k \neq 0$.

In the first case, if $k = 0$, then we wish to prove that $k \cdot f(x) = 0$, the zero function, has limit 0. So, given $\varepsilon > 0$, let $\delta = 1$ (or any other positive number, really). Thus, for all $0 < |x - a| < \delta$, we have that

$$|k \cdot f(x) - kL| = |0 - 0| = 0 < \varepsilon.$$

Thus, $\lim\limits_{x\to a} k \cdot f(x) = k \cdot L$.

Next, assume that $k \neq 0$. Let $\varepsilon > 0$. Since $\lim_{x\to a} f(x) = L$, there exists a δ such that for all x satisfying $0 < |x - a| < \delta$,

$$|f(x) - L| < \frac{\varepsilon}{|k|}.$$

Thus, $|k| \cdot |f(x) - L| < \varepsilon$ and thus $|k \cdot f(x) - k \cdot L| < \varepsilon$. So $\lim\limits_{x\to a} k \cdot f(x) = k \cdot L$.

(b) Let $\varepsilon > 0$. Since $\lim_{x \to a} f(x) = L$, there exists a $\delta_f > 0$ such that for all x satisfying $0 < |x - a| < \delta_f$, we have

$$|f(x) - L| < \frac{\varepsilon}{2}.$$

Similarly, since $\lim_{x \to a} g(x) = M$, there exists a $\delta_g > 0$ such that for all x satisfying $0 < |x - a| < \delta_g$, we have

$$|g(x) - M| < \frac{\varepsilon}{2}.$$

Choose $\delta = \min\{\delta_f, \delta_g\} > 0$. Then, for all x satisfying $0 < |x - a| < \delta$, we have that

$$|(f(x) + g(x)) - (L + M)| = |f(x) - L + g(x) - M| \leq |f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus,

$$\lim_{x \to a} f(x) + g(x) = L + M.$$