Math 431 - Real Analysis I
Solutions to Homework due October 1

In class, we learned of the concept of an *open cover* of a set $S \subset \mathbb{R}^n$ as a collection $\mathcal{F}$ of open sets such that

$$S \subset \bigcup_{A \in \mathcal{F}} A.$$

We used this concept to define a *compact set* S as in which every infinite cover of S has a finite subcover.

Question 1. Show that the following subsets S are not compact by finding an infinite cover $\mathcal{F}$ that has no finite subcover. Be sure to prove that your infinite cover does indeed have no finite subcover; usually a proof by contradiction is best for these.

(a) $S = (0, 1)$

(b) $S = (0, \infty)$

A complete answer would include the following:

(i) providing the infinite cover $\mathcal{F}$;

(ii) showing that $S \subset \bigcup_{A \in \mathcal{F}} A$;

(iii) showing that $\mathcal{F}$ has no finite subcover. The best way to do this is to assume, to the contrary, that there exists some finite subcover $\mathcal{F}'$. Then, show that $S \not\subset \bigcup_{A \in \mathcal{F}'} A$ by finding an $x \in S$ such that $x \notin \bigcup_{A \in \mathcal{F}'} A$.

Solution 1.

(a) Let $\mathcal{F} = \{(0, 1 - 1/n) \,|\, n \in \mathbb{Z}_+, n \geq 2\}$. First, we will show that

$$(0, 1) \subset \bigcup_{n=2}^{\infty} (0, 1 - 1/n).$$

Let $x \in (0, 1)$; thus $0 < x < 1$. Thus, $1 - x > 0$. By the Archimedean Property, there exists an $n \in \mathbb{Z}_+$ such that $1 < (1 - x)n$, and thus $1 - 1/n > x$. Thus, since $0 < x < 1 - 1/n$, $x \in (0, 1 - 1/n)$. Thus,

$$x \in \bigcup_{n=2}^{\infty} (0, 1 - 1/n).$$

Next, assume, to the contrary, that $\mathcal{F}$ has a finite subcover $\mathcal{F}'$. We will find an $x \in (0, 1)$ such that is not covered by $\mathcal{F}'$. Since $\mathcal{F}'$ is a finite subcover, there exists a largest n such that $(0, 1 - 1/n) \in \mathcal{F}'$. Since n is the largest such integer, them for all over $(0, 1 - 1/m) \in \mathcal{F}'$, $m \leq n$. Since $m \leq n$, then $1 - 1/m < 1 - 1/n$. Thus, $(0, 1 - 1/m) \subset (0, 1 - 1/n)$. So,

$$\bigcup_{A \in \mathcal{F}'} A \subset (0, 1 - 1/n).$$

However, $1 - 1/(n + 1) \in (0, 1)$ but $1 - 1/(n + 1) \notin (0, 1 - 1/n)$. Thus, $\mathcal{F}'$ does not cover $(0, 1)$. $\square$

(b) Let $\mathcal{F} = \{(n,(n+2)) \,|\, a \in \mathbb{N}\}$. First we will show that

$$(0,\infty) \subset \bigcup_{n=0}^{\infty} (n, n+2).$$

Let $x \in (0,\infty)$. Then, $x > 0$. Since $\mathbb{Z}_+$ is unbounded, there exists some $k \in \mathbb{Z}_+$ such that $x < k$. Of all of these k, choose the smallest such k (one exists by properties of $\mathbb{Z}_+$). Then, since $k \in \mathbb{Z}_+$, $k-1 \in \mathbb{N}$. Since k is the smallest $k \in \mathbb{Z}$ such that $x < k$, then $k - 1 < x$. Furthermore since $x < k$, then $x < k+1$. Thus,

$$x \in (k-1, k+1) \subset \bigcup_{n=1}^{\infty} (n, n+2).$$

Next, we show that this cover $\mathcal{F}$ has no finite subcover $\mathcal{F}'$. Assume, to the contrary that it does. Then, by finiteness, there is some largest n such that $(n, n+2) \in \mathcal{F}'$. Consider the real number $n+3$. Notice that $n+3 > 0$ and thus $n+3 \in (0,\infty)$. If $n+3$ were still covered by $\mathcal{F}'$, then there would exist some $k \in \mathbb{N}$ such that $n+3 \in (k, k+2)$. Thus, $n+3 < k+2$ and thus, $n < k-1 < k$, contradicting that n is the largest such integer such that $(n, n+2)$ covers S. $\square$

Question 2. Let S be a *discrete set* of $\mathbb{R}^n$. Show that S is compact if and only if S is finite. Note: The direction "if S is finite, then S is compact" does not use the fact that S is discrete; it's true for general finite sets. In proving "If S is infinite, then S is non-compact," you will have to produce an infinite cover of S that has no finite subcover; in this direction, discreteness if necessary.

Solution 2. First, we prove "If S if finite, then S is compact." Write $S = \{x_1, x_2, \ldots x_n\}$. Let $\mathcal{F}$ be a cover of S. Thus, $x_i \in \bigcup_{A \in \mathcal{F}} A$. Thus, there exists some $A_i \in \mathcal{F}$ such that $x_i \in A_i$. Choosing one A_i for each x_i produces at most n open sets. Define $\mathcal{F}'$ to be the finite subcollection $\mathcal{F}' = \{A_1, \ldots, A_n\}$. Since each $x_i \in A_i$, we know that

$$S \subset \bigcup_{i=1}^{n} A_i.$$

Thus, S is compact.

Conversely, assume that S is infinite. We will show that S is non-compact. Since S is discrete, each $x \in S$ is isolated. Thus, for each $x \in S$, there exists an $\varepsilon_x > 0$ such that $B(x;\varepsilon_x) \cap S = \{x\}$. Since each of these balls is open, we can define the open cover

$$\mathcal{F} = \{B(x;\varepsilon_x)) \,|\, x \in S\}.$$

Since S is infinite, this $\mathcal{F}$ is an *infinite* open cover. We will show that $\mathcal{F}$ has no finite subcover. Assume $\mathcal{F}'$ is a finite subcollection of $\mathcal{F}'$. Then, since $\mathcal{F}$ is infinite, there is at least one (in fact, infinitely many) $y \in S$ such that $B(y;\varepsilon_y) \notin \mathcal{F}'$. Since for all $x \in S$, $B(x;\varepsilon_x) \cap S = \{x\}$, the only open set in $\mathcal{F}$ that contains y is $B(y;\varepsilon_y)$. Thus, removing it means that $y \notin \bigcup_{A \in \mathcal{F}'}$. Thus, $\mathcal{F}'$ is not a subcover. Thus, $\mathcal{F}$ is an open cover of S with no finite subcover. Thus, S is not compact. $\square$

Question 3. Prove the following theorem about compacts sets in $\mathbb{R}^n$..

(a) Show that a finite union of compact sets is compact.

(b) Let S be compact and T be closed. Show that $S \cap T$ is compact.

(c) Use (b) to quickly show that a closed subset of a compact set is compact.

(d) Show that the intersection of arbitrarily many compact sets is compact.

Solution 3.

(a) We prove this using the definition of compactness. Let $A_1, A_2, \dots A_n$ be compact sets. Consider the union $\bigcup_{k=1}^{n} A_k$. We will show that this union is also compact. To this end, assume that $\mathcal{F}$ is an open cover for $\bigcup_{k=1}^{n} A_k$. Since $A_i \subset \bigcup_{k=1}^{n} A_k$, then $\mathcal{F}$ is also a cover for A_i. By compactness of A_i, there exists a finite subcover $\mathcal{F}_i'$ for A_i. Consider $\mathcal{F}' = \bigcup_{i=1}^{n} \mathcal{F}_i'$. Since each $\mathcal{F}_i$ is finite, and there are only finitely many such i, $\mathcal{F}'$ is finite as well. Furthermore, since each $\mathcal{F}_i'$ cover A_i, then $\mathcal{F}'$ covers $\bigcup_{k=1}^{n} A_k$. Thus, we have found a finite subcover for the union of our compact sets. Thus, it is compact.

(b) Since S is compact, it is closed and bounded by the Heine-Borel Theorem. Since T is also closed, then $S \cap T$ is closed. Since $S \cap T \subset S$ and S is bounded, $S \cap T$ is also bounded. Thus, $S \cap T$ is a closed and bounded set and thus compact.

(c) Since S be compact and $T \subset S$ be a closed subset of $\mathbb{R}^n$¿ Then, $S \cap T = T$. By (b), $S \cap T = T$ must be compact. Thus, T is compact.

(d) Let $\mathcal{G}$ be a collection of compact sets. We will show that $\bigcap_{A \in \mathcal{G}} A$ is compact. Note that every $A \in \mathcal{G}$ is compact, and thus closed and bounded. Since the intersection of arbitrarily many closed sets is closed, $\bigcap_{A \in \mathcal{G}} A$ is also closed. Choose any $A' \in \mathcal{G}$. Since $\bigcap_{A \in \mathcal{G}} A \subset A'$ and A' is bounded, then $\bigcap_{A \in \mathcal{G}} A$ is bounded. Thus, $\bigcap_{A \in \mathcal{G}} A$ is closed and bounded; thus it is compact.

In class, we learned that a *metric space* is a set M along with a distance function d from $M \times M$ to $\mathbb{R}$ satisfying the following properties for all $x, y, z \in M$:

(i) Positive-definite: $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.

(ii) Symmetry: $d(x, y) = d(y, x)$

(iii) Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

Question 4. Show that the following sets and distance functions d are indeed metric spaces by verifying that they satisfy the three metric space properties.

(a) $M = \mathbb{R}_+$ with distance function $d(x, y) = |\log(x/y)|$.

(b) $M = \mathbb{R}^2$ with its L^1 distance function

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

Solution 4.

(a) Positive-definite: Since the absolute value function is always non-negative, we have that

$$d(x, y) = |\log(x/y)| \geq 0.$$

If $x = y$, then $d(x, x) = |\log(x/x)| = \log 1 = 0$. Lastly, assume that $d(x, y) = 0$. Then, $|\log(x/y)| = 0$ and thus $\log(x/y) = 0$. The only way for this to occur is if $x/y = 1$, which is equivalent to $x = y$.

Symmetry:

$$d(x,y) = |\log(x/y)| = |\log x - \log y| = |-1|\,|\log y - \log x| = |\log(y/x)| = d(y,x).$$

Triangle Inequality: Notice that

$$d(x,z) = |\log(x/z)| = |\log x - \log z|.$$

Adding and subtracting $\log y$ and using the regular triangle inequality, we get that

$$d(x,z) = |\log x - \log y + \log y - \log z| \le |\log x - \log y| + |\log y - \log z| = d(x,y) + d(y,z).$$

(b) Positive-definite: Since $|x_1 - x_2| \ge 0$ and $|y_1 - y_2| \ge 0$, then

$$d((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2| \ge 0.$$

Now, assume that $(x_1,y_1) = (x_2,y_2)$. Then, $|x_1 - x_2| = 0 = |y_1 - y_2|$. Thus,

$$d((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2| = 0.$$

Now, assume that $d((x_1,y_1),(x_2,y_2)) = 0$. Then, $|x_1 - x_2| + |y_1 - y_2| = 0$. Since $|x_1 - x_2| \ge 0$ and $|y_1 - y_2| \ge 0$, then it must be true that $|x_1 - y_1| = 0$ and $|x_2 - y_2| = 0$. Thus, $x_1 = x_2$ and $y_1 = y_2$. So, $(x_1,y_1) = (x_2,y_2)$.

Symmetry: Note that

$$d((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2| = |x_2 - x_1| + |y_2 - y_2| = d((x_2,y_2),(x_1,y_1)).$$

Triangle Inequality: We begin with

$$d((x_1,y_1),(x_3,y_3)) = |x_1 - x_3| + |y_1 - y_3|.$$

If we add and subtract x_2 and y_2 into the two absolute values and apply the usual triangle inequality, we get

$$\begin{aligned} d((x_1,y_1),(x_3,y_3)) &= |(x_1 - x_2) + (x_2 - x_3)| + |(y_1 - y_2) + (y_2 - y_3)|. \\ &\le |x_1 - x_2| + |x_2 - x_3| + |y_1 - y_2| + |y_2 - y_3| \\ &= |x_1 - x_2| + |y_1 - y_2| + |x_2 - x_3| + |y_2 - y_3| \\ &= d((x_1,y_1),(x_2,y_2)) + d((x_2,y_2),(x_3,y_3)) \end{aligned}$$

Question 5. Let M be a non-empty set with metric d. Thus, d satisfies the three metric properties. Let $k > 0$ and consider the new distance function d' given by

$$d'(x,y) = k \cdot d(x,y).$$

Show that d' is also a metric on M by showing it satisfies the three metric properties.

Solution 5. First, we show that d' is positive definite. Since d is positive definite, then $d(x,y) \ge 0$ for all $x, y \in M$. Multiplying by $k > 0$, we get that

$$d'(x,y) = k \cdot d(x,y) \ge 0.$$

Since $k \ge 0$ and $d(x,y) = 0$ if and only if $x = y$, then $d'(x,y) = k \cdot d(x,y) = 0$ if and only if $x = y$.

Next, we show that d' is symmetric. Since d is symmetric, we have that $d(x, y) = d(y, x)$. Thus,

$$d'(x, y) = k \cdot d(x, y) = k \cdot d(y, x) = d'(y, x).$$

Last, we show that d' satisfies the triangle inequality. Since d is a metric, it satisfies a triangle inequality. So,

$$d(x, z) \leq d(x, y) + d(y, z).$$

Since $k > 0$, we can multiply through to get

$$d'(x, z) = k \cdot d(x, z) \leq k \cdot d(x, y) + k \cdot d(y, z) = d'(x, y) + d'(y, z).$$

Satisfying all three metric properties, we have that d' is a metric on M. □

Question 6. Let M be a non-empty set with two metrics d_1 and d_2. Thus, d_1 and d_2 both satisfy the three metric properties. Consider the new distance function d' given by

$$d'(x, y) = d_1(x, y) + d_2(x, y).$$

Show that d' is also a metric on M by showing that it satisfies the three metric properties.

Solution 6. First, we show that d' is non-negative. Since $d_1(x, y), d_2(x, y) \geq 0$, then the sum of two non-negative numbers is non-negative. So,

$$d'(x, y) = d_1(x, y) + d_2(x, y) \geq 0.$$

Since d_1 and d_2 are individually metrics, they have the property that $d_1(x, y) = d_2(x, y) = 0$ if and only if $x = y$. Thus, if $x = y$, then

$$d'(x, y) = d'(x, x) = d_1(x, x) + d_2(x, x) = 0.$$

Conversely, if $d'(x, y) = 0$, then $d_1(x, y) + d_2(x, y) = 0$. Since each term is non-negative, the only way for this to occur is if both d_1 and d_2 are zero. However, this occurs if and only if $x = y$.

Next, we will show d' is symmetric. Since both d_1 and d_2 are symmetric, then $d_1(x, y) = d_1(y, x)$ and $d_2(x, y) = d_2(y, x)$. Thus,

$$d'(x, y) = d_1(x, y) + d_2(x, y) = d_1(y, x) + d_2(y, x) = d'(y, x).$$

Lastly, we show the triangle inequality holds for d'. Since it holds for d_1 and d_2, we have that the following two inequalities hold:

$$d_1(x, z) \leq d_1(x, y) + d_1(y, z)$$
$$d_2(x, z) \leq d_2(x, y) + d_2(y, z).$$

Adding the two inequalities, we get that

$$d'(x, z) = d_1(x, z) + d_2(x, z) \leq d_1(x, y) + d_1(y, z) + d_2(x, y) + d_2(y, z) = d'(x, y) + d'(y, z).$$

Thus the triangle inequality holds.

Satisfying all three metric properties, we have that d' is a metric on M. □