Math 431 - Real Analysis I
Solutions to Homework due November 21

Question 1. The following questions use the ever-important Mean Value Theorem.

(a) Let $f(x)$ be any quadratic polynomial $f(x) = \alpha x^2 + \beta x + \gamma$. Consider the secant line joining the points $(t_1, f(t_1))$ and $(t_2, f(t_2))$. What is the slope of this secant line (in terms of α, β, γ, and t_i)? Simplify as much as possible.

(b) For the f in (a), the Mean Value Theorem guarantees the existence of some $c \in (t_1, t_2)$ such that $f'(c)$ is equal to the above slope. For this particular f, what is this point c?

(c) Use the Mean Value Theorem to deduce the following inequality for all x, y:

$$|\sin y - \sin x| \leq |y - x|.$$

Solution 1.

(a) The slope of the second line joining these points is given by

$$\frac{f(t_1) - f(t_2)}{t_1 - t_2} = \frac{\alpha t_1^2 + \beta t_1 + \gamma - (\alpha t_2^2 + \beta t_2 + \gamma)}{t_1 - t_2} = \frac{\alpha(t_1^2 - t_2^2) + \beta(t_1 - t_2)}{t_1 - t_2} =$$

$$\frac{(t_1 - t_2)[\alpha(t_1 + t_2) + \beta]}{t_1 - t_2} = \alpha(t_1 + t_2) + \beta.$$

(b) Taking the derivative, we have that

$$f'(x) = 2\alpha x + \beta.$$

This will equal our mean slope when

$$2\alpha x + \beta = \alpha(t_1 + t_2) + \beta,$$

which occurs when $x = \dfrac{t_1 + t_2}{2}$, the midpoint of t_1 and t_2.

(c) Consider $f(x) = \sin x$. For any $x < y \in \mathbb{R}$, we have that there exists some c such tath $x < c < y$ such that

$$f'(c) = \frac{\sin y - \sin x}{y - x}.$$

Since the derivative of $\sin x$ is $\cos x$, then $|f'(c)| \leq 1$. Thus, we have that

$$\left| \frac{\sin y - \sin x}{y - x} \right| \leq 1.$$

Cross-multiplying, we get that $|\sin y - \sin x| \leq |y - x|$.

Question 2. Let f be a function that is continuous on $[a, b]$ and second differentiable (i.e., f'' exists) on (a, b). Assume that the line segment joining the points $A = (a, f(a))$ and $B = (b, f(b))$ intersect the graph of f in a third point different from A and B. Show that $f''(c) = 0$ for some $c \in (a, b)$.

Solution 2. We will use the MVT thrice. First, label the point where the secant line intersect the graph as $D = (d, f(d))$. Then, notice that the slope of the secant line from A to B is the same as the slope of the secant line from A to D and from D to B. Thus,

$$\frac{f(b) - f(a)}{b - a} = \frac{f(d) - f(a)}{d - a} = \frac{f(b) - f(d)}{b - d}.$$

Using the MVT on $[a, d]$, we get that there exists some $\alpha \in (a, d)$ such that

$$f'(\alpha) = \frac{f(d) - f(a)}{d - a}$$

Similarly, using the MVT on $[d, b]$, there exists some $\beta \in [d, b]$ such that

$$f'(\beta) = \frac{f(d) - f(b)}{d - b}.$$

Because these two secant slopes are equal, we have that

$$f'(\alpha) = f'(\beta).$$

Now, we can use MVT on the interval $[\alpha, \beta]$ with the differentiable function $f'(x)$. Doing so, we get that there exists some $c \in [\alpha, \beta]$ such that

$$f''(c) = \frac{f'(\alpha) - f'(\beta)}{\alpha - \beta} = 0.$$

Thus, $f''(c) = 0$ as desired.

Question 3. Let f and g be differentiable functions. Show that if $f'(x) = g'(x)$ for all x, then $f(x) = g(x)+k$ where $k \in \mathbb{R}$.

Solution 3. Consider the function $f(x) - g(x)$, which is also differentiable. Notice that its derivative is $f'(x) - g'(x) = 0$. Thus, by a theorem in class, $f(x) - g(x)$ is constant. So, $f(x) - g(x) = k$ for some $k \in \mathbb{R}$. Thus, $f(x) = g(x) + k$.

Question 4. The hypotheses of the Mean Value Theorem are each quite important. They state that f must be continuous on $[a, b]$ and differentiable on (a, b).

(a) Find a counterexample to the MVT if the hypothesis "f is differentiable on (a, b)" is dropped. To do this, find a function that is continuous on $[a, b]$ but not differentiable on (a, b) where

$$f'(c) \neq \frac{f(b) - f(a)}{b - a}$$

for all c.

(b) Find a counterexample to the MVT if the hypothesis "f is continuous on $[a, b]$" is dropped. To do this, find a function that is not continuous on all of $[a, b]$ but f is differentiable on (a, b) where

$$f'(c) \neq \frac{f(b) - f(a)}{b - a}$$

for all c.

Solution 4.

(a) Consider the function $f(x) = |x|$, which is continuous on the closed interval $[-1, 1]$, but is not differentiable at 0. Notice that

$$\frac{f(-1) - f(1)}{-1 - 1} = 0.$$

However, at any $c \in [-1, 1]$ where f' does exists, the derivative is always ± 1, but never 0.

(b) Consider the function defined on $[0, 1]$ given by

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } 0 < x \leq 1 \end{cases}.$$

Notice that f is discontinuous only at $x = 0$. On $(0, 1)$, $f'(x) = 0$ since it is constant on that open interval. However, the mean slope is given by

$$\frac{f(1) - f(1)}{1 - 0} = -1.$$

Thus, there is no c such that $f'(c)$ is equal to the mean slope.

Question 5. Let $a, r \in \mathbb{R}$ with $r \neq 1$. *Use induction* to show that

$$\sum_{k=0}^{n} ar^k = \frac{a - ar^{n+1}}{1 - r}$$

for all $n \geq 0$.

Solution 5. Let $A(n)$ be the statement that

$$\sum_{k=0}^{n} ar^k = \frac{a - ar^{n+1}}{1 - r}.$$

We will show that $A(n)$ is true for all $n \geq 0$. For the base case, notice that

$$\sum_{k=0}^{0} ar^k = ar^0 = a = a \cdot \frac{1 - r}{1 - r} = \frac{a - ar^{0+1}}{1 - r}.$$

Thus, $A(0)$ holds.

Now, assume that $A(n)$ hold for some $n \geq 0$. We will show that $A(n + 1)$ also holds. Starting with the left-hand side of the $A(n + 1)$ expression, we have that

$$\sum_{k=0}^{n+1} ar^k = \sum_{k=0}^{n} ar^k + ar^{n+1} = \frac{a - ar^{n+1}}{1 - r} + ar^{n+1} =$$

$$\frac{a - ar^{n+1}}{1 - r} + ar^{n+1}\frac{1 - r}{1 - r} = \frac{a - ar^{n+1}}{1 - r} + \frac{ar^{n+1} - ar^{n+2}}{1 - r} = \frac{a - ar^{(n+1)+1}}{1 - r}.$$

Thus, $A(n + 1)$ is true. So, by induction, $A(n)$ holds for all $n \geq 0$.

Question 6. In this question, we will show that if $|r| < 1$, then $r^n \to 0$.

(a) State the binomial theorem. Use it to show that if $b > 0$, then $(1 + b)^n > nb$.

(b) Prove that if $|r| < 1$, then $r^n \to 0$ using an $\varepsilon - N$ proof. To do so, it would be wise to note that if $|r| < 1$, then

$$|r| = \frac{1}{1 + b}$$

for some $b > 0$.

Solution 6.

(a) The binomial theorem states that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Thus, for $(1+b)^n$, we have that

$$(1+b)^n = 1 + nb + \binom{n}{2} b^2 + \cdots + b^n.$$

Since $b > 0$, all the terms are positive. Thus, the sum is greater than or equal to each individual term and thus $(1+b)^n > nb$.

(b) Let $\varepsilon > 0$. Since $|r| < 1$, then we can write it as

$$|r| = \frac{1}{1+b}$$

for some $b > 0$. Consider $N = \dfrac{1}{\varepsilon b} > 0$. Assume that $n > \varepsilon = \dfrac{1}{\varepsilon b}$. Then, we have that

$$|r^n - 0| = |r|^n = \frac{1}{(1+b)^n}.$$

Since $(1+b)^n > nb$, we have that

$$\frac{1}{(1+b)^n} < \frac{1}{nb}.$$

Since $n > \frac{1}{\varepsilon b}$, we have that

$$\frac{1}{nb} < \varepsilon.$$

Thus, $|r^n - 0| < \varepsilon$. So, we have that $r^n \to 0$.