Math 431 - Real Analysis I
Homework due November 14

Let S and T be metric spaces. We say that a function $f : S \to T$ is *uniformly continuous* on $A \subset S$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $x, y \in A$ with $d_S(x, y) < \delta$, then $d_T(f(x), f(y)) < \varepsilon$.

Question 1. Let $f : \mathbb{R} \to \mathbb{R}$ be uniform continuous on a set $A \subset \mathbb{R}$.

(a) Let $k \in \mathbb{R}$. Show that $k \cdot f$ is also continuous on A.

(b) If $g : \mathbb{R} \to \mathbb{R}$ is also uniformly continuous on A, show that $f + g$ is uniformly continuous on A.

(c) Let $m, b \in \mathbb{R}$. Show that $h(x) = mx + b$ is uniformly continuou on any $A \subset \mathbb{R}$.

Solution 1.

(a) Let $\varepsilon > 0$. Since f is uniformly continuous on A, there exists a $\delta > 0$ such that for all $x, y \in A$ satisfying $|x - y| < \delta$, then $|f(x) - f(y)| < \frac{\varepsilon}{|k|+1}$. Thus, for all $x, y \in A$ such that $|x - y| < \delta$, we have that

$$|k \cdot f(x) - k \cdot f(y)| = |k||f(x) - f(y)| < \frac{|k|}{|k| + 1}\varepsilon < \varepsilon.$$

Thus, $k \cdot f$ is uniformly continuous on A.

(b) Let $\varepsilon > 0$. Since f is uniformly continuous on A, there exists a $\delta_f > 0$ such that whenever $x, y \in A$ such that $|x - y| < \delta_f$, then $|f(x) - f(y)| < \dfrac{\varepsilon}{2}$. Similarly, since g is uniformly continuous on A, there exists a δ_g such that whenever $x, y \in A$ such that $|x - y| < \delta_g$, then $|g(x) - g(y)| < \dfrac{\varepsilon}{2}$. Let $\delta = \min\{\delta_f, \delta_g\}$. Then, for all $x, y \in A$ satisfying that $|x - y| < \delta$, we have that

$$|f(x) + g(x) - (f(y) + g(y))| \le |f(x) - f(y)| + |g(x) - g(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $f + g$ is uniformly continuous on A.

(c) Let $\varepsilon > 0$. Let $\delta = \dfrac{\varepsilon}{|m| + 1} > 0$. Let $x, y \in A$ such that $|x - y| < \delta$, we have that

$$|f(x) - f(y)| = |(mx + b) - (my + b)| = |mx - my| = |m||x - y| < \frac{|m|}{|m| + 1}\varepsilon < \varepsilon.$$

Thus, f is uniformly continuous on any A.

Question 2. Show that the function $f(x) = x^n$ uniformly continuous on $[-1, 1]$ for all $n \in \mathbb{Z}_+$. To do so, it may be helpful to remember that we previously proved that

$$x^n - y^n = (x - y)\sum_{k=0}^{n-1} x^k y^{n-1-k}.$$

Solution 2. Let $\varepsilon > 0$. Let $\delta = \dfrac{\varepsilon}{n} > 0$. Let $x, y \in [0, 1]$ such that $|x - y| < \delta = \dfrac{\varepsilon}{n}$. For these x, y, we have that

$$|f(x) - f(y)| = |x^n - y^n| = \left|(x - y)\sum_{k=0}^{n-1} x^k y^{n-1-k}\right| = |(x - y)| \left|\sum_{k=0}^{n-1} x^k y^{n-1-k}\right| \le$$

$$|x-y|\sum_{k=0}^{n-1}|x^k y^{n-1-k}| = |x-y|\sum_{k=0}^{n-1}|x|^k|y|^{n-1-k}.$$

Since $x, y \in [0,1]$, we have that $|x|, |y| \leq 1$. Thus,

$$\sum_{k=0}^{n-1}|x|^k|y|^{n-1-k} \leq \sum_{k=0}^{n-1} 1 \cdot 1 = n.$$

Continuing our computation, we have that

$$|x-y|\sum_{k=0}^{n-1}|x|^k|y|^{n-1-k} \leq |x-y| \cdot n < \frac{\varepsilon}{n} \cdot n = \varepsilon.$$

Thus, for any $x, y \in [0,1]$ satisfying $|x-y| < \delta$, we have that $|f(x) - f(y)| < \varepsilon$.
In class, we gave the definition of the derivative of a function at a point. If f is a real function defined on some open interval (a,b) such that $c \in (a,b)$, then we say f is *differentiable at* c of the following limit exists:

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}.$$

If this limit exists, then we denote it by $f'(c)$ as call it the *derivative of* f *at* c.

Question 3. Use the limit definition to compute the derivative of

$$f(x) = \frac{3x+4}{2x-1}$$

at every $c \neq 1/2$.

Solution 3. Computing, we have

$$f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c} = \lim_{x\to c}\frac{\frac{3x+4}{2x-1} - \frac{3c+4}{2c-1}}{x-c} =$$

$$\lim_{x\to c}\frac{\frac{(3x+4)(2c-1)-(3c+4)(2x-1)}{(2x-1)(2c-1)}}{x-c} = \lim_{x\to c}\frac{\frac{5(x-c)}{(2x-1)(2c-1)}}{x-c} =$$

$$\lim_{x\to c}\frac{5}{(2x-1)(2c-1)} = \frac{5}{2c-1}.$$

Question 4. Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0 \end{cases}$$

Show that f is differentiable at 0 by showing that $f'(0) = 0$. To do so, you will have to use the limit definition of the derivative, which will include an $\varepsilon - \delta$ proof.

Solution 4. We will show that $f'(0) = 0$ by showing that

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = 0.$$

Since $f(0) = 0$, this is equivalent to showing that $\lim_{x\to 0}\frac{f(x)}{x} = 0$. Let $\varepsilon > 0$. Choose $\delta = \varepsilon > 0$. We will show that whenever $0 < |x-0| < \delta$, then $\left|\frac{f(x)}{x} - 0\right| < \varepsilon$. We consider the following two cases: $x < 0$ or $x > 0$. If $x < 0$, then $f(x) = 0$. Thus, $\left|\frac{f(x)}{x} - f(0)\right| = |0-0| < \varepsilon$. If $x > 0$, then $f(x) = x^2$. Thus,

$$\left|\frac{f(x)}{x} - 0\right| = \left|\frac{x^2}{x} - 0\right| = |x-0| < \delta = \varepsilon.$$

Thus,

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$$

and thus $f'(0) = 0$.

Question 5. In this question, we will prove the quotient rule using the product rule and the chain rule.

(a) Use the definition of the derivative to show that if $f(x) = \frac{1}{x}$, then

$$f'(a) = \frac{-1}{a^2}.$$

(b) Use (a), the product rule, and the chain rule to prove the quotient rule.

Solution 5.

(a) Computing the derivative, we have

$$f'(a) = \lim_{x\to a} \frac{\frac{1}{x}-\frac{1}{a}}{x-a} = \lim_{x\to a} \frac{\frac{a-x}{xa}}{x-a} = \lim_{x\to a} \frac{-1}{xa} = -\frac{1}{a^2}.$$

(b) Consider the quotient $\frac{f(x)}{g(x)}$, which we will write as

$$f(x)\cdot [g(x)]^{-1}.$$

By the chain rule and (a), we have that the derivative of $[g(x)]^{-1}$ is $-\frac{g'(a)}{g^2(a)}$. Thus, by the product rule, we have that

$$\left(\frac{f}{g}\right)'(a) = \left(f(x)\cdot[g(x)]^{-1}\right)'(a) = f'(a)[g(a)]^{-1} + f(a)\cdot\frac{-g'(a)}{g^2(a)} =$$
$$\frac{f(a)g'(a)-f(a)g'(a)}{g^2(a)}.$$

Question 6.

(a) Consider the function

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that f is not differentiable at $x = 0$. [Hint: Differentiable implies continuous]

(b) Consider the function

$$g(x) = \begin{cases} x^2\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that g is differentiable at 0 at that $g'(0) = 0$.

Solution 5.

(a) Consider the sequence $x_n = \frac{1}{\pi/2 + 2\pi n}$. Notice that $x_n \to 0$. However,

$$f(x_n) = \sin(\pi/2 + 2\pi n) = 1 \to 1 \neq 0 = f(0).$$

Thus, f is not continuous at 0 and is therefore non-differentiable at 0.

(b) We will show that

$$\lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = 0.$$

Notice that $g(x) = 0$ and when $x \neq 0$, $g(x) = x^2 \sin(1/x)$. Thus, we wish to show that

$$\lim_{x \to 0} \frac{x^2 \sin(1/x)}{x} = \lim_{x \to 0} x \sin(1/x) = 0.$$

Let $\varepsilon > 0$. Set $\delta = \varepsilon$. Then, assume that x satisfies that $0 < |x - 0| < \delta = \varepsilon$. Then,

$$|x \sin(1/x) - 0| = |x|| \sin(1/x)| \leq |x| < \delta = \varepsilon.$$

Thus, $g'(0) = 0$.