Math 431 - Real Analysis
Homework due September 5

Question 1. Let $a, b \in \mathbb{R}$.

(a) Show that if $a + b$ is rational, then a is rational or b is irrational.

(b) Use (a) to show that if $a + b$ is rational, then a and b are both rational or both irrational.

In class on Monday, we learned of boundedness, the supremum/infimum, and the Completeness Axiom. Given a bounded set $S \subset \mathbb{R}$, a number b is called a *supremum* or *least upper bound* for S if the following hold:

(i) b is an upper bound for S, and

(ii) if c is an upper bound for S, then $b \leq c$.

Similarly, given a bounded set $S \subset \mathbb{R}$, a number b is called an *infimum* or *greatest lower bound* for S if the following hold:

(i) b is a lower bound for S, and

(ii) if c is a lower bound for S, then $c \leq b$.

If b is a supremum for S, we write that $b = \sup S$. If it is an infimum, we write that $b = \inf S$.

We were also introduced to our tenth and final axiom, the *Completeness Axiom*. This axiom states that any non-empty set $S \subset \mathbb{R}$ that is bounded above has a supremum; in other words, if S is a non-empty set of real numbers that is bounded above, there exists a $b \in \mathbb{R}$ such that $b = \sup S$.

Question 2. Show that if a set $S \subset \mathbb{R}$ has a supremum, then it is unique. Thus, we can talk about *the* supremum of a set, instead of the *a* supremum of a set.

Question 3. Let S be a non-empty subset of $\mathbb{R}$.

(a) Let $-S = \{-x \in \mathbb{R} \mid x \in S\}$. Show that S has a supremum b if and only if $-S$ has an infimum $-b$.

(b) Use (a) to show that if T is a non-empty set that is bounded below, then T has an infimum.

Question 4. Prove the following *Comparison Theorem*: Let $S, T \subset \mathbb{R}$ be non-empty sets such that $s \leq t$ for every $s \in S$ and $t \in T$. If T has a supremum, then so does S and,

$$\sup S \leq \sup T.$$

Question 5. Consider the set

$$S = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{Z}_+ \right\}.$$

(a) Show that $\max S = 1$.

(b) Show that if d is a lower bound for S, then $d \leq 0$. [Hint: A proof by contradiction might be helpful, as well as the Archimedean Property.]

(c) Use (b) to show that $0 = \inf S$.

Question 6. Consider the set

$$T = \left\{ (-1)^n \left(1 - \frac{1}{n}\right) \,\middle|\, n \in \mathbb{Z}_+ \right\}.$$

(a) Show that 1 is an upper bound for T.

(b) Similar to 5b, show that if d is an upper bound for T, then $d \geq 1$.

(c) Use (a) and (b) to show that $\sup T = 1$.