# Math 431 - Real Analysis
## Homework due September 17

In class, we learned of the famous *Cauchy-Schwarz Inequality.* Given two $n$-vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, Cauchy-Schwarz relates its dot product with the norms of the individual vectors:

$$(\mathbf{x} \cdot \mathbf{y})^2 \leq \|\mathbf{x}\|^2 \|\mathbf{y}\|^2.$$

Written component-wise with

$$\mathbf{x} = (x_1, x_2, \ldots, x_n) \text{ and } \mathbf{y} = (y_1, y_2, \cdots, y_n),$$

the Cauchy-Schwarz inequality is

$$\left( \sum_{k=1}^{n} x_k y_k \right)^2 \leq \left( \sum_{k=1}^{n} x_k^2 \right) \left( \sum_{k=1}^{n} y_k^2 \right).$$

**Question 1.** Many times, the Cauchy-Schwarz Inequality can be used to obtain some interesting inequalities by simply choosing an appropriate vector $\mathbf{x}$ and $\mathbf{y}$.

(a) Let $a, b, c \in \mathbb{R}$. Show that

$$(a + b + c)^2 \leq 3(a^2 + b^2 + c^2).$$

(b) Let $a, b, c \in \mathbb{R}_+$. Show that

$$(a + b + c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 9.$$

(c) Let $a_1, a_2, \cdots a_n \in \mathbb{R}$. Show the *Sum of Squares inequality:*

$$\left( \frac{1}{n} \sum_{k=1}^{n} a_k \right)^2 \leq \frac{1}{n} \sum_{k=1}^{n} a_k^2.$$

In class on Friday, we learned of several new definitions that will help us to describe the topology of $\mathbb{R}^n$. The first was an *open ball of radius $r$ about* $\mathbf{x}$, which was given by

$$B(\mathbf{x}; r) = \{\mathbf{y} \in \mathbb{R}^n \mid \|\mathbf{y} - \mathbf{x}\| < r\}.$$

If $S \subset \mathbb{R}^n$, a point $\mathbf{x}$ is called an *interior point* of $S$ if there exists an $\varepsilon > 0$ such that $B(\mathbf{x}; \varepsilon) \subset S$. The set of all interior points of $S$ is denoted by int $S$, and it is always true that int $S \subset S$.

If $S =$ int $S$, then we say that $S$ is an *open set.* In other words, $S$ is open if and only if for every $\mathbf{x} \in S$, there exists an $\varepsilon > 0$ such that $B(\mathbf{x}; \varepsilon) \subset S$.

**Question 2.** It is often easier to prove that a given set $S$ is *not open.* To do so, one needs to find a point $\mathbf{x} \in S$ such that for no $r > 0$, $B(\mathbf{x}; r) \subset S$. In other words, one needs to find a $\mathbf{x} \in S$ such that for all $r > 0$, there exists some $y \in B(\mathbf{x}; r)$ such that $y \in B(\mathbf{x}; r)$ but $y \notin S$. Show that the following subsets $S \subset \mathbb{R}^n$ are *not open.*

(a) $\{a\} \subset \mathbb{R}$

(b) $\{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$

(c) $\{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ and } y \geq 0\} \subset \mathbb{R}^2$

**Question 3.** In what follows, we will demonstrate an important topological property of $\mathbb{Q} \subset \mathbb{R}$.

(a) Let $a \in \mathbb{Q}$. Show that $a + \frac{\sqrt{2}}{n}$ is irrational for all $n \in \mathbb{Z}_+$.

(b) Use (a) to show that $\mathbb{Q}$ is not an open subset of $\mathbb{R}$.

Given a set $S \subset \mathbb{R}^n$, a point $\mathbf{x} \in S$ is called an *isolated point of* $S$ if there exists an $\varepsilon > 0$ such that $B(\mathbf{x}; \varepsilon) \cap S = \{x\}$. In other words, $\mathbf{x}$ is *isolated* in $S$ if there is a small enough $\varepsilon > 0$ such that $B(\mathbf{x}; \varepsilon)$ intersects $S$ only at $\mathbf{x}$ itself. A set $S$ is called *discrete* if every point in $S$ is isolated.

**Question 4.** Show that the following sets are or are not discrete.

(a) Show that $\mathbb{Z}$ is a discrete subset of $\mathbb{R}$.

(b) Show that ever finite subset of $\mathbb{R}$ is a discrete subset of $\mathbb{R}$.

(c) Show that $S = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{Z}_+ \right\}$ is a discrete subset of $\mathbb{R}$

(d) Show that $T = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{Z}_+ \right\} \cup \{0\}$ is not a discrete subset of $\mathbb{R}$.

**Question 5.** Let $U, V \in \mathbb{R}$ be open sets. Consider the product set

$$U \times V = \{(x, y) \,|\, x \in U, y \in V\} \subset \mathbb{R}^2.$$

Show that $U \times V$ is open by showing that each $(x, y) \in U \times V$ is an interior point.

**Question 6.** Consider the set

$$T = \{\mathbf{x} \in \mathbb{R}^2 \,|\, \|\mathbf{x}\| < 1\}.$$

Geometrically, this set is just an "open disk" of radius 1 about the origin. Consider

$$\mathbb{S}^1 = \{\mathbf{x} \in \mathbb{R}^2 \,|\, \|\mathbf{x}\| = 1\}.$$

Geometrically, $\mathbb{S}^1$ is the circle of radius 1 about the origin. We will show that every point in $\mathbb{S}^1$ is an accumulation point of $T$ (and therefore an adherent point of $T$).

As a hint, you may want to follow something similar to the below outline:

Let $\mathbf{x} \in \mathbb{S}^1$. We will show that for all $\varepsilon > 0$, $B(\mathbf{x}; \varepsilon) \cap (T - \{\mathbf{x}\}) \neq \varnothing$. First, note that $T - \{x\} = T$ since $\mathbf{x} \notin T$. Thus, we wish to show that $B(\mathbf{x}; \varepsilon) \cap T \neq \varnothing$. Then, consider the 2 cases: $\varepsilon > 1$ or $0 < \varepsilon \leq 1$. In the last case, it might be wise to consider $\left(1 - \dfrac{\varepsilon}{2}\right)\mathbf{x}$.