Math 431 - Real Analysis I
Homework due October 8

Question 1. Recall that any set M can be given the discrete metric d_d given by

$$d_d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

For the below, let M be any set with the discrete metric.

(a) Show that *any* subset S of M is an open set.

(b) Use (a) to show that any subset of M is closed.

(c) Show that any subset S of M is discrete (hence the name 'discrete metric').

(d) Show that a subset S of M is compact if and only if it is finite.

Question 2. Use the $\varepsilon - N$ definition of the convergence of a sequence to show that following sequences converge to the indicated limits. Unless otherwise stated, all sequences are valid for $n \geq 1$.

(a) $\dfrac{1}{n^2} \to 0$

(b) $\dfrac{2}{\sqrt{n}} + 1 \to 1$

(c) $e^{-n} \to 0$

Question 3. In what follows, let M be a metric space with metric d.

(a) A sequence $\{x_n\}$ in a metric space is called *eventually constant* if there exists some N such that for all $n > N$, $x_n = p$ for some $p \in M$. Show that any eventually constant sequence converges.

(b) Let $k \in \mathbb{R}$ and let $\{x_n\}$ be a real sequence. Show that if $x_n \to a$, then the sequence $\{k \cdot x_n\}$ converges to $k \cdot a$.

Question 4. In this question, we will investigate specific examples of convergence in the metric space

$$C([0,1]) = \{f : [0,1] \to \mathbb{R} \,|\, f \text{ is continuous}\}.$$

In other words, $C([0,1])$ is the set of all continuous real-valued functions whose domain is $[0,1]$. We will consider $C([0,1])$ with its L^1 metric given by

$$d(f,g) = \int_0^1 |f(x) - g(x)|\, dx.$$

Below, you will be given a sequence of these functions. You should

(i) Draw/graph these functions for $n = 2, 3, 5$, and 10.

(ii) Show that f_n converges to the indicated function.

(a) Let $\{f_n\}$ be the sequence given by $f_n(x) = x^n$. Show that f_n converges to the constant 0 function.

(b) Let $\{f_n\}$ be the sequence given by

$$f_n(x) = \begin{cases} nx & \text{if } 0 \leq x \leq 1/n \\ 1 & \text{if } 1/n < x \leq 1 \end{cases}$$

Show that f_n converges to the constant function 1.

Question 5. In Question 3a, we showed that in any metric space, every eventually constant sequence converges. In this question, we will show that in a metric space with the discrete metric, the converse is true. In other words, let M be a metric space with the discrete metric d_d (defined in Question 1). Show that if x_n converges, then $\{x_n\}$ is eventually constant.

Question 6. Let $\{x_n\}$ be a real sequence. We will show that $x_n \to 0$ if and only if $|x_n| \to 0$.

(a) Show that

$$-|x_n| \leq x_n \leq |x_n|$$

for all n.

(b) Use (a) to show that $|x_n| \to 0$, then $x_n \to 0$.

(c) Use an $\varepsilon - N$ proof to show that if $x_n \to 0$, then $|x_n| \to 0$.

(d) Conclude that $x_n \to 0$ if and only if $|x_n| \to 0$.