Math 431 - Real Analysis I
Homework due October 31

In class, we learned that a function $f : S \to T$ between metric spaces (S, d_S) and (T, d_T) is continuous if and only if the pre-image of every open set in T is open in S. In other words, f is continuous if for all open $U \subset T$, the pre-image $f^{-1}(U) \subset S$ is open in S.

Question 1. Let S, T, and R be metric spaces and let $f : S \to T$ and $g : T \to R$. We can define the composition function $g \circ f : S \to R$ by

$$g \circ f(s) = g(f(s)).$$

(a) Let $U \subset R$. Show that $(g \circ f)^{-1}(U) = f^{-1}\left(g^{-1}(U)\right)$

(b) Use (a) to show that if f and g are continuous, then the composition $g \circ f$ is also continuous

Question 2. Let (S, d_S) and (T, d_T) be metric spaces and let $f : S \to T$.

(a) A function is called *constant* if $f(s) = t_0$ for all $s \in S$. Show that any constant function is continuous.

(b) Show that if d_S is the discrete metric, then any function f is continuous.

Question 3. The *floor function* $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = \lfloor x \rfloor$, where $\lfloor x \rfloor x$ is the largest integer less than or equal to x.

(a) Let $a \notin \mathbb{Z}$. Use an $\varepsilon - \delta$ proof to show that $f(x) = \lfloor x \rfloor$ is continuous at a.

(b) Let $a \in \mathbb{Z}$. Show that $f(x) = \lfloor x \rfloor$ is not continuous at a. To do so, find an $\varepsilon > 0$ such that for any $\delta > 0$, there exists an x with $|x - a| < \delta$ such that $|f(x) - f(a)| \geq \varepsilon$.

Question 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.

(a) Assume that $f(x) \geq 0$ for all $x \in [0, 1]$. Show that if $f(c) > 0$ for some $c \in (0, 1)$, then

$$\int_0^1 f(x)\, dx > 0.$$

(b) Show that the above is no longer true if the term "continuous" is dropped. That is, given an example of a (necessarily discontinuous) function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \geq 0$ and $f(c) > 0$ for some $c \in (0, 1)$, yet

$$\int_0^1 f(x)\, dx = 0.$$

Question 5. Recall that we can equip $C([0,1])$, the space of all continuous functions on $[0,1]$, with its L^1 metric, which is given by

$$d(f,g) = \int_0^1 |f(x) - g(x)|\, dx.$$

Consider the function $\varphi : C([0,1]) \to \mathbb{R}$ given by

$$\varphi(f) = \int_0^1 f(x)\, dx.$$

In this question, we will show that φ is a continuous function.

(a) Show that

$$\left| \int_0^1 h(x)\, dx \right| \leq \int_0^1 |h(x)|\, dx.$$

Hint: We previously proved that $-|a| \leq a \leq |a|$ for all $a \in \mathbb{R}$.

(b) Use the above to give an $\varepsilon - \delta$ proof that φ is continuous.

Question 6. In this question, we will prove the *Brouwer Fixed Point Theorem.* In what follows, let $f : [0,1] \to [0,1]$ be a continuous function.

(a) A number $c \in [0,1]$ is called a *fixed point of f* if $f(c) = c$. Define $g(x) = f(x) - x$. Show that c is a fixed point of f if and only if c is a root of g.

(b) Use (a) and Bolzano's Theorem to show that every continuous function $f : [0,1] \to [0,1]$ has a fixed point.

(c) Give an example of a continuous function $f : (0,1) \to (0,1)$ that has *no fixed points.* Be sure to prove that our f is fixed-point-free.

Question 7. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

We will show that f is continuous only at $a = 0$.

(a) Use an $\varepsilon - \delta$ proof to show that $f(x)$ is continuous at $a = 0$.

(b) Use the theorem relating convergent sequences to continuous functions to show that if $a \neq 0$, then $f(x)$ is not continuous at a.