Math 431 - Real Analysis I
Homework due October 22

Question 1. Sequences are frequently given *recursively*, where a beginning term x_1 is specified and subsequent terms can be found using a recursive relation. One such example is the sequence defined by $x_1 = 1$ and

$$x_{n+1} = \sqrt{2 + x_n}.$$

(a) For $n = 1, 2, \ldots, 10$, compute x_n. A calculator may be helpful.

(b) Show that x_n is a monotone increasing sequence. A proof by induction might be easiest.

(c) Show that the sequence x_n is bounded below by 1 and above by 2. For the above bound, induction might again be best.

(d) Use (b) and (c) to conclude that x_n converges.

Question 2. One very important class of sequences are *series*, in which we add up the terms of a given sequence. One such example is the following sequence:

$$S_n = \sum_{k=0}^{n} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}.$$

(a) For $n = 0, 1, 2, \ldots, 6$, compute S_n. Again, a calculator may be helpful; be sure to use several digits.

(b) Show that S_n is monotone increasing.

(c) Use induction to show that for all $n \geq 1$, $n! \geq 2^{n-1}$.

(d) Use (c) to show that

$$S_n \leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}}.$$

(e) Use well-known facts from Calculus II and the geometric series to show that

$$1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} < 3.$$

(f) Use (b), (d), and (e) to conclude that S_n converges.

Question 3. In class, we learned that a sequence in $\mathbb{R}^k$ is convergent if and only if it is Cauchy. We have previously proven using the definition of convergence that the sequence

$$x_n = \frac{1}{n}$$

converges (to 0). Thus, it should also be Cauchy. In this problem, we will prove directly that it is Cauchy.

(a) Let $n, m \in \mathbb{Z}_+$. Show that

$$\left| \frac{1}{n} - \frac{1}{m} \right| < \frac{1}{n} + \frac{1}{m}.$$

(b) Use (a) to show that $x_n = \frac{1}{n}$ is a Cauchy sequence. To do so, given an $\varepsilon > 0$, find an N such that for all $n, m > N$,

$$\left| \frac{1}{n} - \frac{1}{m} \right| < \frac{1}{n} + \frac{1}{m} < \varepsilon.$$

Question 4. Consider the sequence of partial sums given by

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2}.$$

We will show that S_n converges by showing it is Cauchy.

(a) If $n, m \in \mathbb{Z}_+$ with $m > n$, show that

$$|S_m - S_n| = \sum_{k=n+1}^{m} \frac{1}{k^2}.$$

(b) Show that $\frac{1}{k^2} < \frac{1}{k(k-1)}$ for $k \geq 2$.

(c) Show that

$$\sum_{k=n+1}^{m} \frac{1}{k(k-1)} = \frac{1}{n} - \frac{1}{m}.$$

As a hint, think about *telescoping series* from Calculus II.

(d) Use the above to show that

$$|S_m - S_n| < \frac{1}{m} + \frac{1}{n}.$$

(e) Use (d) in a proof to show that S_n is Cauchy and thus converges.

Question 5. Let $f, g : \mathbb{R} \to \mathbb{R}$ with $a, L, M, k \in \mathbb{R}$. Furthermore, assume that

$$\lim_{x \to a} f(x) = L \quad \text{and} \quad \lim_{x \to a} g(x) = M.$$

Give an $\varepsilon - \delta$ proof to show the following:

(a) $\lim_{x \to a} k \cdot f(x) = k \cdot L$

(b) $\lim_{x \to a} f(x) + g(x) = L + M$