In class, we learned of the concept of an open cover of a set \(S \subset \mathbb{R}^n\) as a collection \(\mathcal{F}\) of open sets such that 
\[
S \subset \bigcup_{A \in \mathcal{F}} A.
\]

We used this concept to define a compact set \(S\) as in which every infinite cover of \(S\) has a finite subcover.

**Question 1.** Show that the following subsets \(S\) are not compact by finding an infinite cover \(\mathcal{F}\) that has no finite subcover. Be sure to prove that your infinite cover does indeed have no finite subcover; usually a proof by contradiction is best for these.

(a) \(S = (0, 1)\)

(b) \(S = (0, \infty)\)

A complete answer would include the following:

(i) providing the infinite cover \(\mathcal{F}\);

(ii) showing that \(S \subset \bigcup_{A \in \mathcal{F}} A\);

(iii) showing that \(\mathcal{F}\) has no finite subcover. The best way to do this is to assume, to the contrary, that there exists some finite subcover \(\mathcal{F}'\). Then, show that \(S \not\subset \bigcup_{A \in \mathcal{F}'} A\) by finding an \(x \in S\) such that \(x \not\in \bigcup_{A \in \mathcal{F}'} A\).

**Question 2.** Let \(S\) be a discrete set of \(\mathbb{R}^n\). Show that \(S\) is compact if and only if \(S\) is finite. Note: The direction “if \(S\) is finite, then \(S\) is compact” does not use the fact that \(S\) is discrete; it’s true for general finite sets. In proving “If \(S\) is infinite, then \(S\) is non-compact,” you will have to produce an infinite cover of \(S\) that has no finite subcover; in this direction, discreteness if necessary.

**Question 3.** Prove the following facts about compact sets in \(\mathbb{R}^n\).

(a) Show that a finite union of compact sets is compact.

(b) Let \(S\) be compact and \(T\) be closed. Show that \(S \cap T\) is compact.

(c) Use (b) to quickly show that a closed subset of a compact set is compact.

(d) Show that the intersection of arbitrarily many compact sets is compact.

In class, we learned that a metric space is a set \(M\) along with a distance function \(d\) from \(M \times M\) to \(\mathbb{R}\) satisfying the following properties for all \(x, y, z \in M\):

(i) **Positive-definite:** \(d(x, y) \geq 0\) and \(d(x, y) = 0\) if and only if \(x = y\).

(ii) **Symmetry:** \(d(x, y) = d(y, x)\)

(iii) **Triangle Inequality:** \(d(x, z) \leq d(x, y) + d(y, z)\).

**Question 4.** Show that the following sets and distance functions \(d\) are indeed metric spaces by verifying that they satisfy the three metric space properties.
(a) $M = \mathbb{R}_+$ with distance function $d(x, y) = |\log(x/y)|$.

(b) $M = \mathbb{R}^2$ with its $L^1$ distance function

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

**Question 5.** Let $M$ be a non-empty set with metric $d$. Thus, $d$ satisfies the three metric properties. Let $k > 0$ and consider the new distance function $d'$ given by

$$d'(x, y) = k \cdot d(x, y).$$

Show that $d'$ is also a metric on $M$ by showing it satisfies the three metric properties.

**Question 6.** Let $M$ be a non-empty set with two metrics $d_1$ and $d_2$. Thus, $d_1$ and $d_2$ both satisfy the three metric properties. Consider the new distance function $d'$ given by

$$d'(x, y) = d_1(x, y) + d_2(x, y).$$

Show that $d'$ is also a metric on $M$ by showing that it satisfies the three metric properties.