Math 431 - Real Analysis I
Homework due October 1

In class, we learned of the concept of an *open cover* of a set $S \subset \mathbb{R}^n$ as a collection $\mathcal{F}$ of open sets such that

$$S \subset \bigcup_{A \in \mathcal{F}} A.$$

We used this concept to define a *compact set* S as in which every infinite cover of S has a finite subcover.

Question 1. Show that the following subsets S are not compact by finding an infinite cover $\mathcal{F}$ that has no finite subcover. Be sure to prove that your infinite cover does indeed have no finite subcover; usually a proof by contradiction is best for these.

(a) $S = (0, 1)$

(b) $S = (0, \infty)$

A complete answer would include the following:

(i) providing the infinite cover $\mathcal{F}$;

(ii) showing that $S \subset \bigcup_{A \in \mathcal{F}} A$;

(iii) showing that $\mathcal{F}$ has no finite subcover. The best way to do this is to assume, to the contrary, that there exists some finite subcover $\mathcal{F}'$. Then, show that $S \not\subset \bigcup_{A \in \mathcal{F}'} A$ by finding an $x \in S$ such that $x \notin \bigcup_{A \in \mathcal{F}'} A$.

Question 2. Let S be a *discrete set* of $\mathbb{R}^n$. Show that S is compact if and only if S is finite. Note: The direction "if S is finite, then S is compact" does not use the fact that S is discrete; it's true for general finite sets. In proving "If S is infinite, then S is non-compact," you will have to produce an infinite cover of S that has no finite subcover; in this direction, discreteness if necessary.

Question 3. Prove the following facts about compacts sets in $\mathbb{R}^n$.

(a) Show that a finite union of compact sets is compact.

(b) Let S be compact and T be closed. Show that $S \cap T$ is compact.

(c) Use (b) to quickly show that a closed subset of a compact set is compact.

(d) Show that the intersection of arbitrarily many compact sets is compact.

In class, we learned that a *metric space* is a set M along with a distance function d from $M \times M$ to $\mathbb{R}$ satisfying the following properties for all $x, y, z \in M$:

(i) Positive-definite: $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.

(ii) Symmetry: $d(x, y) = d(y, x)$

(iii) Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

Question 4. Show that the following sets and distance functions d are indeed metric spaces by verifying that they satisfy the three metric space properties.

(a) $M = \mathbb{R}_+$ with distance function $d(x, y) = |\log(x/y)|$.

(b) $M = \mathbb{R}^2$ with its L^1 distance function

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

Question 5. Let M be a non-empty set with metric d. Thus, d satisfies the three metric properties. Let $k > 0$ and consider the new distance function d' given by

$$d'(x, y) = k \cdot d(x, y).$$

Show that d' is also a metric on M by showing it satisfies the three metric properties.

Question 6. Let M be a non-empty set with two metrics d_1 and d_2. Thus, d_1 and d_2 both satisfy the three metric properties. Consider the new distance function d' given by

$$d'(x, y) = d_1(x, y) + d_2(x, y).$$

Show that d' is also a metric on M by showing that it satisfies the three metric properties.