Math 431 - Real Analysis I
Homework due November 21

Question 1. The following questions use the ever-important Mean Value Theorem.

(a) Let $f(x)$ be any quadratic polynomial $f(x) = \alpha x^2 + \beta x + \gamma$. Consider the secant line joining the points $(t_1, f(t_1))$ and $(t_2, f(t_2))$. What is the slope of this secant line (in terms of α, β, γ, and t_i)? Simplify as much as possible.

(b) For the f in (a), the Mean Value Theorem guarantees the existence of some $c \in (t_1, t_2)$ such that $f'(c)$ is equal to the above slope. For this particular f, what is this point c?

(c) Use the Mean Value Theorem to deduce the following inequality for all x, y:

$$|\sin y - \sin x| \leq |y - x|.$$

You may use the fact that $\sin x$ is everywhere differentiable.

Question 2. Let f be a function that is continuous on $[a, b]$ and second differentiable (i.e., f'' exists) on (a, b). Assume that the line segment joining the points $A = (a, f(a))$ and $B = (b, f(b))$ intersect the graph of f in a third point different from A and B. Show that $f''(c) = 0$ for some $c \in (a, b)$.

Question 3. Let f and g be differentiable functions. Show that if $f'(x) = g'(x)$ for all x, then $f(x) = g(x)+k$ where $k \in \mathbb{R}$.

Question 4. The hypotheses of the Mean Value Theorem are each quite important. They state that f must be continuous on $[a, b]$ and differentiable on (a, b).

(a) Find a counterexample to the MVT if the hypothesis "f is differentiable on (a, b)" is dropped. To do this, find a function that is continuous on $[a, b]$ but not differentiable on (a, b) where

$$f'(c) \neq \frac{f(b) - f(a)}{b - a}$$

for all c.

(b) Find a counterexample to the MVT if the hypothesis "f is continuous on $[a, b]$" is dropped. To do this, find a function that is not continuous on all of $[a, b]$ but f is differentiable on (a, b) where

$$f'(c) \neq \frac{f(b) - f(a)}{b - a}$$

for all c.

Question 5. Let $a, r \in \mathbb{R}$ with $r \neq 1$. *Use induction* to show that

$$\sum_{k=0}^{n} ar^k = \frac{a - ar^{n+1}}{1 - r}$$

for all $n \geq 0$.

Question 6. In this question, we will show that if $|r| < 1$, then $r^n \to 0$.

(a) State the binomial theorem. Use it to show that if $b > 0$, then $(1 + b)^n > nb$.

(b) Prove that if $|r| < 1$, then $r^n \to 0$ using an $\varepsilon - N$ proof. To do so, it would be wise to note that if $|r| < 1$, then

$$|r| = \frac{1}{1 + b}$$

for some $b > 0$.