Question 1. The following questions use the ever-important Mean Value Theorem.

(a) Let \( f(x) \) be any quadratic polynomial \( f(x) = \alpha x^2 + \beta x + \gamma \). Consider the secant line joining the points \( (t_1, f(t_1)) \) and \( (t_2, f(t_2)) \). What is the slope of this secant line (in terms of \( \alpha, \beta, \gamma \), and \( t_i \))? Simplify as much as possible.

(b) For the \( f \) in (a), the Mean Value Theorem guarantees the existence of some \( c \in (t_1, t_2) \) such that \( f'(c) \) is equal to the above slope. For this particular \( f \), what is this point \( c \)?

(c) Use the Mean Value Theorem to deduce the following inequality for all \( x, y \):

\[ |\sin y - \sin x| \leq |y - x|. \]

You may use the fact that \( \sin x \) is everywhere differentiable.

Question 2. Let \( f \) be a function that is continuous on \([a,b]\) and second differentiable (i.e., \( f'' \) exists) on \((a,b)\). Assume that the line segment joining the points \( A = (a, f(a)) \) and \( B = (b, f(b)) \) intersect the graph of \( f \) in a third point different from \( A \) and \( B \). Show that \( f''(c) = 0 \) for some \( c \in (a,b) \).

Question 3. Let \( f \) and \( g \) be differentiable functions. Show that if \( f'(x) = g'(x) \) for all \( x \), then \( f(x) = g(x) + k \) where \( k \in \mathbb{R} \).

Question 4. The hypotheses of the Mean Value Theorem are each quite important. They state that \( f \) must be continuous on \([a, b]\) and differentiable on \((a, b)\).

(a) Find a counterexample to the MVT if the hypothesis “\( f \) is differentiable on \((a, b)\)” is dropped. To do this, find a function that is continuous on \([a, b]\) but not differentiable on \((a, b)\) where

\[ f'(c) \neq \frac{f(b) - f(a)}{b - a} \]

for all \( c \).

(b) Find a counterexample to the MVT if the hypothesis “\( f \) is continuous on \([a, b]\)” is dropped. To do this, find a function that is not continuous on all of \([a, b]\) but \( f \) is differentiable on \((a, b)\) where

\[ f'(c) \neq \frac{f(b) - f(a)}{b - a} \]

for all \( c \).

Question 5. Let \( a, r \in \mathbb{R} \) with \( r \neq 1 \). Use induction to show that

\[ \sum_{k=0}^{n} ar^k = \frac{a - ar^{n+1}}{1 - r} \]

for all \( n \geq 0 \).

Question 6. In this question, we will show that if \( |r| < 1 \), then \( r^n \to 0 \).

(a) State the binomial theorem. Use it to show that if \( b > 0 \), then \( (1 + b)^n > nb \).

(b) Prove that if \( |r| < 1 \), then \( r^n \to 0 \) using an \( \varepsilon - N \) proof. To do so, it would be wise to note that if \( |r| < 1 \), then

\[ |r| = \frac{1}{1 + b} \]

for some \( b > 0 \).