Math 431 - Real Analysis I
Homework due November 14

Let S and T be metric spaces. We say that a function $f : S \to T$ is *uniformly continuous* on $A \subset S$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $x, y \in A$ with $d_S(x, y) < \delta$, then $d_T(f(x), f(y)) < \varepsilon$.

Question 1. Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on a set $A \subset \mathbb{R}$.

(a) Let $k \in \mathbb{R}$. Show that $k \cdot f$ is also continuous on A.

(b) If $g : \mathbb{R} \to \mathbb{R}$ is also uniformly continuous on A, show that $f + g$ is uniformly continuous on A.

(c) Let $m, b \in \mathbb{R}$. Show that $h(x) = mx + b$ is uniformly continuou on any $A \subset \mathbb{R}$.

Question 2. Use the $\varepsilon-\delta$ definition to show that the function $f(x) = x^n$ uniformly continuous on $[-1, 1]$ for all $n \in \mathbb{Z}_+$. To do so, it may be helpful to remember that we previously proved that

$$x^n - y^n = (x - y) \sum_{k=0}^{n-1} x^k y^{n-1-k}.$$

In class, we gave the definition of the derivative of a function at a point. If f is a real function defined on some open interval (a, b) such that $c \in (a, b)$, then we say f is *differentiable at* c of the following limit exists:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

If this limit exists, then we denote it by $f'(c)$ as call it the *derivative of* f *at* c.

Question 3. Use the limit definition to compute the derivative of

$$f(x) = \frac{3x + 4}{2x - 1}$$

at every $c \neq 1/2$.

Question 4. Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0 \end{cases}$$

Show that f is differentiable at 0 by showing that $f'(0) = 0$. To do so, you will have to use the limit definition of the derivative, which will include an $\varepsilon-\delta$ proof.

Question 5. In this question, we will prove the quotient rule using the product rule and the chain rule.

(a) Use the definition of the derivative to show that if $f(x) = \dfrac{1}{x}$, then

$$f'(a) = \frac{-1}{a^2}.$$

(b) Use (a), the product rule, and the chain rule to prove the quotient rule.

Question 6.

(a) Consider the function

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that f is not differentiable at $x = 0$. [Hint: Differentiable implies continuous]

(b) Consider the function

$$g(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that g is differentiable at 0 at that $g'(0) = 0$.