Math 431 - Real Analysis I
Homework due December 5

Question 1.

(a) Let $a_k, b_k \geq 0$ for all k. Show that if $\displaystyle\sum_{k=0}^{\infty} a_k$ converges and b_k is a bounded sequence, then $\displaystyle\sum_{k=0}^{\infty} a_k b_k$ converges as well.

(b) Find a counterexample to above statement if the hypothesis "$a_k, b_k \geq 0$" is removed.

Question 2. Show that

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$$

converges if and only if $p > 1$.

Question 3.

(a) Give an example of a divergent series $\sum_{k=1}^{\infty} a_k$ for which $\sum_{k=1}^{\infty} a_k^2$ converges.

(b) Prove that if $a_k \geq 0$ and $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} a_k^2$ converges as well.

(c) Find a counterexample to the above statement if the hypothesis "$a_k \geq 0$" is removed.

Question 4. In this question, we will show that if both

$$\sum_{k=0}^{\infty} a_k^2 \quad \text{and} \quad \sum_{k=0}^{\infty} b_k^2$$

converges, then

$$\sum_{k=0}^{\infty} a_k b_k$$

converges absolutely.

(a) Show that $2|a_k b_k| \leq a_k^2 + b_k^2$ for all k.

(b) Use (a) to show that if $\sum_{k=0}^{\infty} a_k^2$ and $\sum_{k=0}^{\infty} b_k^2$ converges, then $\sum_{k=0}^{\infty} a_k b_k$ converges absolutely.

Question 5. Let a_n be a sequence of non-zero real numbers such that the sequence $\frac{a_{n+1}}{a_n}$ of ratios is a constant sequence. Show that $\sum_{k=1}^{\infty} a_k$ is a geometric series.

In class, we learned of the following definition for the *limit superior* and *limit inferior* of a sequence x_n.

Suppose a number U has the following two properties:

- For all $\varepsilon > 0$, there exists an N such that for all $n > N$, $x_n < U + \varepsilon$; and
- For all $\varepsilon > 0$ and $m > 0$, there exists an $n > m$ such that $x_n > U - \varepsilon$.

Then, $\limsup x_n = U$.

Similarly, suppose that a number L has the following two properties:

- For all $\varepsilon > 0$, there exists an N such that for all $n > N$, $x_n > L - \varepsilon$; and
- For all $\varepsilon > 0$ and $m > 0$, there exists an $n > m$ such that $x_n < L + \varepsilon$.

Then $\liminf x_n = L$.

Question 6. Let x_n be a sequence. If $\liminf x_n = A = \limsup x_n$, show that x_n converges and that, in fact, $x_n \to A$.

Question 7. Consider the series

$$\sum_{k=1}^{\infty} \sqrt{k+1} - \sqrt{k}.$$

(a) Show that

$$\sqrt{k+1} - \sqrt{k} = \frac{1}{\sqrt{k+1} + \sqrt{k}}.$$

(b) Use (a) to decide if the series converges or diverges.

Question 8. Consider the series $\sum_{k=1}^{\infty} a_k$ where

$$a_k = \left(\frac{2}{(-1)^k - 3} \right)^k.$$

(a) Compute $\left| \frac{a_{k+1}}{a_k} \right|$ for general k. Use this to compute

$$\limsup \left| \frac{a_{k+1}}{a_k} \right| \quad \text{and} \quad \liminf \left| \frac{a_{k+1}}{a_k} \right|.$$

Can you use the Ratio Test to reach a conclusion about the convergence or divergence of the series?

(b) Compute $|a_k|^{1/k}$ for general k. Use this to compute

$$\limsup |a_k|^{1/k}.$$

Can you use the Root Test to reach a conclusion about the convergence or divergence of the series?

(c) In spite of the powerful Ratio Test and Root Tests, at times the more rudimentary tests are more helpful. Show that our series diverges using the Divergence Test.