Math 431 - Real Analysis
Homework due August 27

On the first day of class, we learned of nine (of the total of ten) axioms that uniquely define the field of real numbers $\mathbb{R}$. We also saw that many of the basic properties we know to be true of real numbers can be rather easily deduced from these axioms. The purpose of this first question is to familiarize yourself with these types of proofs. Question 1 deals with proving some well-known facts about the $<$ relation. In Question 1, you are free to assume basic arithmetic facts (which can be proven), such as the following:

$$0 \cdot a = 0, \ a \cdot (-b) = -ab, \ -(-a) = a, \ 1 \neq 0, \text{ etc.}$$

In other words, you can assume the basic *algebraic* properties of real numbers, but not properties related to $<$.

Question 1. Use the nine axioms introduced in class to prove the following. Be sure to cite which axioms you are using. In what follows, let $a, b, c \in \mathbb{R}$.

(a) $a > 0$ if and only if $-a < 0$.

(b) $-1 < 0$.

(c) $a > 0$ if and only if $a^{-1} > 0$.

(d) If $a > 0$ and $b < 0$, then $a \cdot b < 0$.

(e) If $a < b$ and $c < 0$, then $c \cdot a > c \cdot b$.

Question 2. Let $n \in \mathbb{Z}$ and $x, y \in \mathbb{R}$ and consider the following expression:

$$(x - y) \sum_{k=0}^{n-1} x^k y^{n-1-k}.$$

(a) For $n = 1, 2, 3$, evaluate the above expression and expand & simplify as much as possible.

(b) Using your observations from (a), conjecture a general pattern.

(c) Prove your conjectured pattern from (b).

In what follows, let $n \in \mathbb{Z}_+$.

Question 3. Show that if $2^n - 1$ is prime, then n is prime. A prime number of the form $2^n - 1$ is called a *Mersenne prime*. Hint: Prove the contrapositive and use your conjectured equation from 2b in your proof.

Question 4. Show that if $2^n + 1$ is prime, then n is a power of 2. A prime number of the form $2^{2^m} + 1$ is called a *Fermat prime*. Hint: As with (3), prove the contrapositive and use your 2b equation.

In class on Monday, we will be formally defining the set of rational numbers $\mathbb{Q}$. We would like to define $\mathbb{Q}$ as the following set:

$$\left\{ \frac{p}{q} \,\middle|\, p, q \in \mathbb{Z}, q \neq 0 \right\}.$$

The problem is, of course, that this definition treats elements like $\frac{3}{6}$ and $\frac{1}{2}$ as unequal. To rectify this, we will offer a more formal definition using the idea of an equivalence relation.

Question 5. Consider the set

$$\mathcal{Q} = \left\{ \frac{p}{q} \,\middle|\, p, q \in \mathbb{Z}, q \neq 0 \right\}.$$

Define a relation $\sim$ on $\mathcal{Q}$ given by

$$\frac{p_1}{q_1} \sim \frac{p_2}{q_2} \text{ if and only if } p_1 q_2 = p_2 q_1.$$

Below, we will show that $\sim$ is an *equivalence relation*, and therefore be able to define the rational numbers $\mathbb{Q}$ as the set of *equivalence classes of elements in* $\mathcal{Q}$.

(a) Reflexivity: Show that $\dfrac{p_1}{q_1} \sim \dfrac{p_1}{q_1}$.

(b) Symmetry: Show that if $\dfrac{p_1}{q_1} \sim \dfrac{p_2}{q_2}$, then $\dfrac{p_2}{q_2} \sim \dfrac{p_1}{q_1}$.

(c) Transitivity: Show that if $\dfrac{p_1}{q_1} \sim \dfrac{p_2}{q_2}$ and $\dfrac{p_2}{q_2} \sim \dfrac{p_3}{q_3}$, then $\dfrac{p_1}{q_1} \sim \dfrac{p_3}{q_3}$.